ISSN 1339-3189

8/2017

Mladá veda

Young Science

Asociácia doktorandov Slovenska

Determinanty pracovnej mobility na Slovensku

Úvod do problematiky nákupných centier

www.mladaveda.sk

Mladá veda
Young Science

MEDZINÁRODNÝ VEDECKÝ ČASOPIS MLADÁ VEDA / YOUNG SCIENCE

Číslo 8, ročník 5., vydané v decembri 2017
ISSN 1339-3189
Kontakt: info@mladaveda.sk, tel.: +421 908 546 716, www.mladaveda.sk
Fotografia na obálke: Pltník v Pieninách. © Branislav A. Švorc, foto.branisko.at

VYDAVATEĽ
Vydavateľstvo UNIVERSUM, spol. s r. o.
www.universum-eu.sk
Javorinská 26, 080 01 Prešov
Slovenská republika

Mladá veda
Young Science
ISSN 1339-3189

Vol. 5 (8), pp. 234-244

EXAMPLE OF MACROSCOPIC MODELING IN HIGH SCHOOLS

PŘÍKLAD MAKROSKOPICKÉHO MODELOVÁNÍ NA STŘEDNÍCH A VYSOKÝCH ŠKOLÁCH

Jana Vysoká[1]

Autorka působí jako asistent na Katedře informatiky a přírodních věd při Vysoké škole technické a ekonomické v Českých Budějovicích. Ve svém výzkumu se věnuje tématice matematického modelování a je autorkou disertační práce s názvem Matematické modelování ve výuce na středních školách.

The author works as a lecturer at the Department of Informatics and Natural Sciences at the Institute of Technology and Business in České Budějovice. In her research she deals with the topic of mathematical modeling and is the author of a dissertation entitled Mathematical Modeling in Teaching in High Schools.

Abstract
This paper offers a demonstration of how some advanced topics discussed at universities can be presented in an accessible way to students in secondary schools or in high schools where mathematical modeling is not part of the lesson. The main selected task is focused on simple example of mathematical modeling in solving simple transport tasks. In this work, we will pay attention to the issue of traffic flow modeling from a macroscopic perspective. First, we derive a mathematical model in the form of partial differential equations and then we will focus on solving this equation using the method of characteristics. The interpretation is presented in a way that should be easily grasped by students.
Key words: mathematical modeling, partial differential equation, method of characteristics

Abstrakt
Tento článek nabízí ukázku toho, jak lze některá témata probíraná na vysokých školách zpřístupnit studentům na středních školách nebo vysokých školách, kde není matematické modelování součástí výuky. Hlavní cíl je zaměřen na jednoduchý příklad matematického modelování při řešení snadné dopravní úlohy. V této práci budeme věnovat pozornost problematice modelování dopravního toku z makroskopického hlediska. Nejprve odvodíme matematický model ve formě parciální diferenciální rovnice a pak se zaměříme na řešení této

[1] Affilation: RNDr. Jana Vysoká, The Institute of Technology and Business in České Budějovice, Okružní 10, 37001 České Budějovice, The Czech republic
E-mail: vysoka@mail.vstecb.cz

rovnice metodou charakteristik. Výklad je prezentován způsobem, s jehož pomocí by studenti mohli problematiku pochopit.
Klíčová slova: Matematické modelování, parciální diferenciální rovnice, metoda charakteristik

Introduction

Mathematical modeling is a constantly evolving modern discipline, which is used in many areas of human activity. Modeling can be used in various disciplines e.g. technical, biological, economical or social. We meet mathematical models every day together with weather forecasting, the modeling of transport networks, population, inventory management, groundwater flow, the determination of flood areas during floods and civil engineering works etc. Models have become an integral part of a tool for forecasting the development of various processes. Using the appropriate mathematical model has many advantages and helps us to understand easier complex phenomena and processes and contexts, allows the simulation of different possible outcomes. Therefore, modeling simple phenomena using procedures and methods deserves to be included into the curricula of selected schools. The effort of some schools is to enhance the teaching the subjects using of commercial programs for example Matlab or Statistics to demonstrate problems in various subjects such as mathematics, physics, biology or chemistry. This material could serve as an expansion of preparation of learning materials designed for university students who have chosen advanced mathematics as an elective, or for high school students who are engaged in mathematics beyond compulsory curriculum. From there it is only a small step to the numerical modeling. The graphs in the paper were constructed using a graphical software Derive 6 (Kutzler, Kokol-Voljc, 2003).

The density of vehicles and the flow of vehicles

The movement of vehicles on the highway can be described analogously with the movement of fluid – a fluid particle motion. The flow of vehicles on the highway can be (from a certain angle and under given conditions) understood as the flow of fluid particles (e.g. when viewed on the road from the aircraft). Therefore it will be necessary to distinguish the concept of a macroscopic speed as the speed of the flow of vehicles and a microscopic speed as the speed of individual vehicles. The motion of fluid particles is influenced by the movement of surrounding particles but the driving of the driver depends on other factors. The driver adjusts the speed accordingly to rational considerations of the situation in front of him or behind him and perceives other factors which may have impact on the speed. The particles of the fluid may collide in its movement with other particles; the driver of the vehicle tries to avoid the collision with another vehicle, of course. Because vehicles represent units in our model, we will assume that they have the identical length. Further we will assume for simplicity that the vehicle traffic will follow only in one direction. In order to determine the number of vehicles on the section of the highway being measured per a given unit of time, we have to take into account, firstly, the overall density of vehicles and, secondly, their speed. The density of vehicles on the monitored section is a concept that would deserve a more detailed analysis. The density of vehicles could be defined as the ratio of the number of vehicles to the unit length of the road and could be will identified by the symbol ρ:

The density of vehicles = the number of vehicles / the unit length of the road.

For example, if the observed road section is 1000 m long and if 35 vehicles are located in this section, and then the density of vehicles is the value $\rho = 0.035$. In the case, that measured road segment is blank, then $\rho = 0$. If $\rho = 1$, then we are talking about a traffic jam. We will determine the density of vehicles in a given place x and in a given time t. We will denote it as a function of two real variables in the form $\rho(x,t) = \rho(x(t))$, where the variable x depends on the variable t.

Another important concept of determining the character of road traffic is the flow of vehicles, which can be defined by this way:

The flow of vehicles = the number of vehicles / the time unit.

The flow of vehicles can be defined in many ways, the simplest one is to define the flow of vehicles as the product of the density and the vehicle speed at which the vehicles are moving. We assume always that the speed of the observed group of vehicles is constant. If we introduce a uniform label for the speed of vehicles as v in the text and we use the symbol φ for the flow of vehicles symbol φ, then yields

$$\varphi(\rho) = v\rho. \tag{1}$$

In the event, that the vehicle speed, will be changed as a result of other conditions such as a traffic accident or a speed limitation the speed or changes in the movement of vehicles joined with light signals the flow function need not be in the form of the linear function in the form (1) and may depend on other factors.

Characteristic of the road traffic

What means a continuous traffic, a limited traffic or a traffic jam in a practice? We will assume that each vehicle has a length of 4.5 meters and drivers try to keep the safe distance (a time lag of 2 seconds). The rule of the distance of 2 seconds means the following situation: Once while driving the vehicle, we will focus on some fixed point in the roadway e.g. a tree or a bollard and from the moment this point misses vehicle in front of you, we deduct 2 seconds. If we miss the same point in less than that, then a controlled vehicle is not in a safe distance from the vehicle ahead. If the vehicle is traveling in the speed of 90 km/h, than it is a distance of approximately 50 meters. In the section of 1000 m long there are a maximum of 18 vehicles in any given time. We can then distinguish the following situations:

1. If the distance between vehicles is 50 m, then on a stretch of road 1000 m long it can move freely from 0 to 18 vehicles. The size of the flow of vehicles will be maximized if the vehicles are not limiting each other during driving. Therefore, the maximum flow of vehicles will be 1,620 vehicles per 1 hour. Whereby, in this case the density is greater the flow is greater. In practice, we are talking about the continuous traffic.

2. If a greater number of vehicles i.e. in the range 19 to 100 vehicles will move on the same road segment, then each vehicle is located in the free section of at least 10 m long. In this case, the driver has to keep an attention to the situation before and after the controlled vehicle and adequately respond to any changes. The speed of vehicles will change in certain sections of road and the traffic can be alternately compressed or diluted. The more is traffic dense, the greater the flow of vehicles is less. The traffic is then characterized as limited.
3. If the monitored section of road contains more than 100 vehicles, than the driver is forced to stop. This then leads to a local traffic jam. If the road contains approximately 200 vehicles, than there is a global jam on the road and all vehicles stand. The flow of vehicles is zero.

We could give illustrative examples of the practice and therefore it will be appropriate to reasonably define the specific form of the function of the flow of vehicles. Suppose that the maximum speed is the same for all vehicles, let it be denoted by the symbol v_{max}. We will omit some reflections on the psychological aspects of driving in a row, for example, if the driver has the vehicle ahead far, then he will accelerate the ride. Based on empirical experience in practice it can be assumed that, if the roadway is empty, therefore the vehicle can move in the maximum speed. Conversely, if the road is full, there is a traffic jam (and the vehicles stop). These properties has for example a function, (Hokr, 2005) in the form

$$v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right). \tag{2}$$

Then we can introduce the flow function in the form

$$\varphi(\rho) = \rho v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right), \tag{3}$$

where yields

$$\varphi'(\rho) = v_{max} - 2\frac{\rho v_{max}}{\rho_{max}}.$$

Macroscopic model of the vehicle flow

Let us monitor the section of the road on which the vehicle moves in the constant speed in the time interval. The density of vehicles ρ in the given segment and time is specified. The basic consideration will be based on the fact that the change of the number of vehicles in the section being measured by the unit of time corresponds to the difference between the number of cars that enter and leave the given segment. We determine the number of vehicles in the time $t = 0$ and in the time $t = t_0 > 0$, their difference will represent a change of the density of vehicles detected in the time $t = t_0$ and we calculate the number of the vehicles in the time t_0 which leave the measured section in the same time – a change of the flow function measured in the given place. If we denote the change of density in the considered time by the symbol $\Delta\rho_t$ and the change of the flow function in the given section $\Delta\rho_x$, where x is a variable describing the location of the vehicle on the highway, then under the previous considerations is

$$\Delta\rho_t + \Delta\rho_x = 0.$$

This relationship is possible to rewrite under certain assumptions in the form of partial differential equations of the first order in the following form

$$\rho_t + v\rho_x = 0. \qquad (4)$$

This equation could be simply verbally explained as follows: The change of the density of vehicles with respect to a certain time interval is identical to the change of the flow function measured in the given section. If the flow function is not given in a linear form, than the equation (4) is changing into the form

$$\rho_t + \varphi_x = 0. \qquad (5)$$

Equation (5) can be rewritten under certain assumptions as

$$\rho_t + [\varphi(\rho)]_x = 0. \qquad (6)$$

We will be able to determine the specific solution of the given task after we will know the distribution of the density of vehicles in a certain place and time. The most common and most natural choice of conditions is the initial distribution of the density of vehicles, therefore the description of the situation in the beginning; let us denote this function by the symbol $\rho(x,0) = \rho_0(x)$. The problem in the form $\rho_t + [\varphi(\rho)]_x = 0$ together with the initial condition $\rho(x,0) = \rho_0(x)$ is called the initial problem (*Cauchy problem*).

The method of characteristics

One of the ways how to describe the solution of the mentioned problem is the way of using the graphical presentation with help of so-called *characteristics*. The equation (4) with the initial condition $\rho(x,0) = \rho_0(x)$ has a simple form, and therefore its solution can be easily guessed. A search function ρ of variables x and $t > 0$ has the form

$$\rho(x,t) = \rho_0(x - vt),$$

This fact can be easily verified by a test. If we are differentiating according to variable x or t, then we are taking the remaining variable as constant and differentiation rules remain unchanged. The function ρ is constant on the all lines with slopes v in the form $x - vt = c$, where c is a real constant. In other words, the density of the given initial conditions is transmitted in a time on a straight line with the slope v or the derivative of the function ρ in

Mladá veda
Young Science
ISSN 1339-3189

Vol. 5 (8), pp. 234-244

the direction v is equal to zero. The intersection of this line and the axis t is denoted by the symbol t_0. These lines, on which density is constant, are called *characteristic lines* or *characteristic* corresponding to the given problem. A drawing characteristics is quite simple and we can get a good idea about solving a given task in a predetermined time and place by the using the graphical representation. The importance of characteristics remains unchanged even when we are solving a nonlinear equation (5). Characteristics are defined as a straight lines with the slope $\varphi'(\rho_0)$. The characteristic line with the slope equal to the value $\varphi'(\rho_0)$ intersects the axis t in the point t_0 and has an equation in the form

$$x = t_0 + \varphi'(\rho_0)t.$$

It is easy to verify by a simply substitution that the given function of the form

$$\rho(x,t) = \rho_0(x - \varphi'(\rho)t)$$

satisfies the equation (5). The value of the solution $\rho(x,t)$ is equal to the corresponding value of the initial conditions $\rho_0(x)$ in the points of this line.

There are some examples of simple situations:

1. Vehicles are going equally in the constant speed, the traffic density is not changed, see fig. 1.

2. Vehicles are not moving and staying in the row, this is an example illustrating the traffic jam, see fig. 2.

3. Vehicles are gradually slowing, their density is increasing, and slopes of characteristics are increasing until there is a traffic jam, see fig. 3.

4. Vehicles are starting to creep, their density is decreasing, slopes of characteristics are decreasing, see fig. 4.

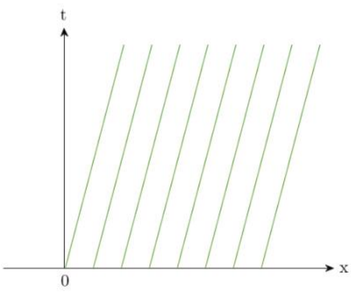


Fig. 1

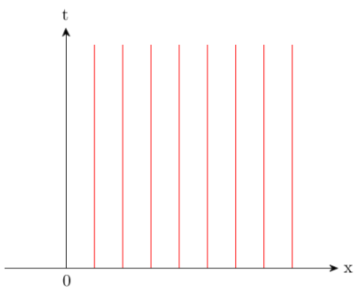
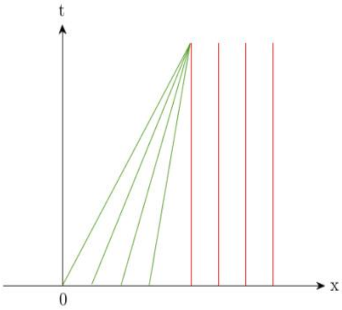


Fig. 2

Mladá veda
Young Science
ISSN 1339-3189
Vol. 5 (8), pp. 234-244



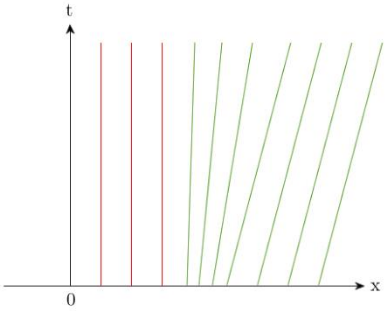


Fig. 3 Fig. 4

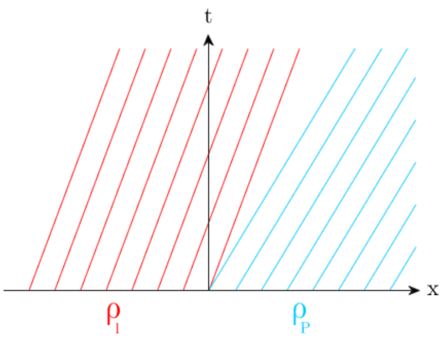
Combining all these examples it is possible to describe the complicated situation as are the traffic directed by traffic lights or the traffic on sections with reduced speeds. To be able to solve complicated cases of initial conditions, it would be advisable to stop at the cases mentioned in examples 3 and 4. Let us consider the task in the form

$$\rho_t + [v\rho]_x = 0 \qquad (7)$$

with the initial condition in the form of the jump $\rho_0(x) = \rho_p$ for $x \geq 0$ and $\rho_0(x) = \rho_l$ for $x < 0$, where ρ_p and ρ_l are given nonnegative real numbers. This problem is called the *Riemann problem*. Then we could distinguish two different situations:

1. $\rho_l < \rho_p$

Characteristics for the value ρ_l are illustrated in the red color and characteristics for the value ρ_p in the blue color, see fig. 5. The figure 5 shows that in the period between red and blue areas (areas defined by inequality $\rho_l t < x < \rho_p t$) characteristics are "missing". If we complete an empty part by lines with slopes of values between ρ_l and ρ_p in the green color, than we get continuous information on the transmitted value of the density in the jump, see fig. 6.



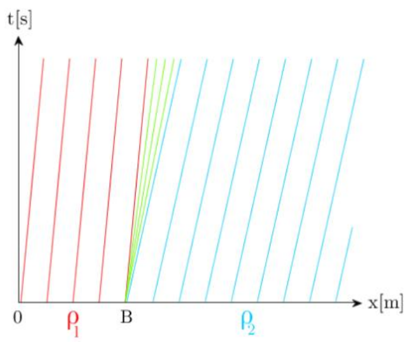


Fig. 5 Fig. 6

1. $\rho_l > \rho_p$

The figure 7 again shows that the characteristics are crossing; it means that we cannot determine a unique value of a density at any time. The direction, in which the information on

Mladá veda
Young Science
ISSN 1339-3189
Vol. 5 (8), pp. 234-244

jump will propagate from the first intersections of the characteristics (so called shock wave), must be only one because the same information is moving in both directions. The slope of this single characteristic, which represents the propagation of shock wave, see fig. 8, is given by so called *Rankine-Hugoniot condition* in the form

$$s = \frac{\varphi(\rho_l) - \varphi(\rho_p)}{\rho_l - \rho_p}. \tag{8}$$

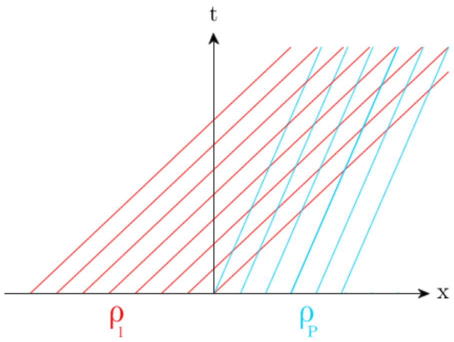


Fig. 7

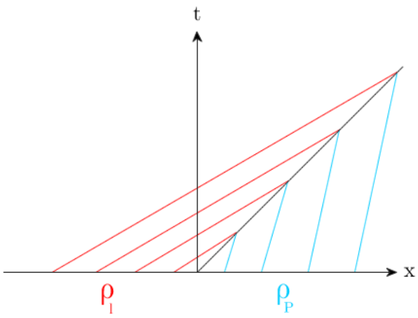


Fig. 8

The derivation of this condition extends beyond the scope of this text but if a student is interested in this topic, we can recommend studying relevant pages in (Drábek, Holubová, 2007) or (Le Veque, 2001, 2002).

Exercises:

Draw the following situations by using characteristics. Monitored section is 1 km long, in the first measuring section there is detected the value of the density of vehicles ρ_1 and in the second section, which is measured from the point B, the value of the density of vehicles ρ_2. Sketch a solution and the point B set in a distance of 100 m in the positive part of the axis x.

1. $\rho_1 = 0.021$, $\rho_2 = 0.2$, $v_{\max} = 14$ m/s

2. $\rho_1 = 0.2$, $\rho_2 = 0.046$, $v_{\max} = 8.3$ m/s

Solution:

1. The density $\rho_1 = 0.021$ represents 21 vehicles per 1 km and vehicles move in the given speed $v_{\max} = 14$ m/s, which is 50 km/h. The density $\rho_2 = 0.2$ represents 200 vehicles per 1 km or the traffic jam, vehicles are moving in the zero velocity. Therefore, it is the first described case, where $\rho_l < \rho_p$. It is necessary to calculate the value of the slopes of characteristics corresponding to the given densities ρ_1 and ρ_2 which means to determine the value according to (8). Substituting given values into a following relationship we will get this result:

$$\varphi'(\rho) = v_{\max} - \frac{2\rho v_{\max}}{\rho_{\max}} = 14 - 2\frac{0.021.14}{0.2} \cong 11,$$

Then we can calculate the direction of propagation of the shock wave as follows

$$s = \frac{0 - 0.26}{0.2 - 0.021} \cong -1.47 .$$

The resulting graph of characteristics is shown in fig. 9.

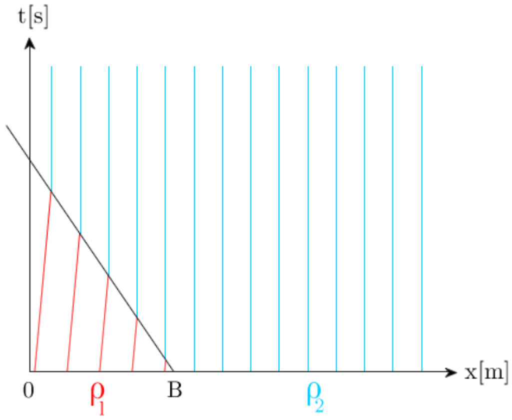


Fig. 9

2. Another example represents a case, where vehicles are in the traffic jam and starting to move taking off at a maximum speed equal to 8.3 m/s = 3 km/h, which is the second described alternative with the condition $\rho_l > \rho_p$. The situation can be illustrated using previous calculations already easily, see fig. 10.

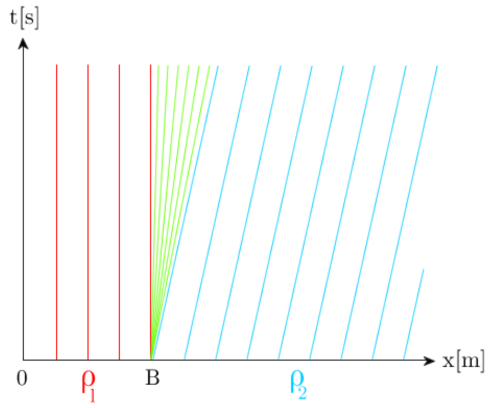


Fig. 10

The initial condition can be defined in a more complex form, where there may be a combination of all the events described above. Monitored section of road with a length of 1000 m is divided into three parts: 0 m - 500 m with a maximum speed of 50 km/ h, vehicles are complying with the safety distance of 2 500 m - 600 m with a maximum density of vehicles simulating a row before traffic lights (maximum speed 0 km/ h) and the section 600 m - 1000 m with a maximum speed of 30 km/ h. The initial condition is specified in the first segment by value ρ_1 = 0.021 vehicles / m (the red color), in the second section of the road ρ_2 = 0.2 vehicles/m (the blue color) and in the last part is the initial density value determined by the value $\rho_3 = 0.046$ vehicles/m (the black color), see fig. 11. In the time $t = 0$ s the traffic light turns to be green and the vehicles start to move. To formulate this problem we have deliberately used the data of the previous examples (Jelínek, Vysoká, 2013, 2014).

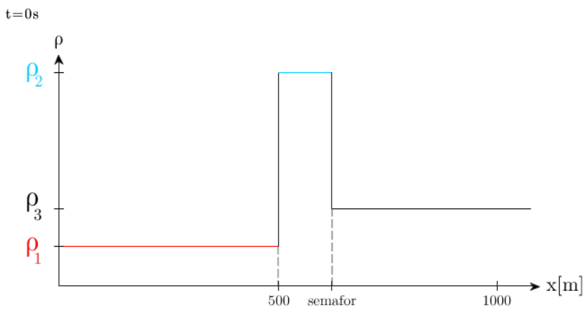


Fig. 11

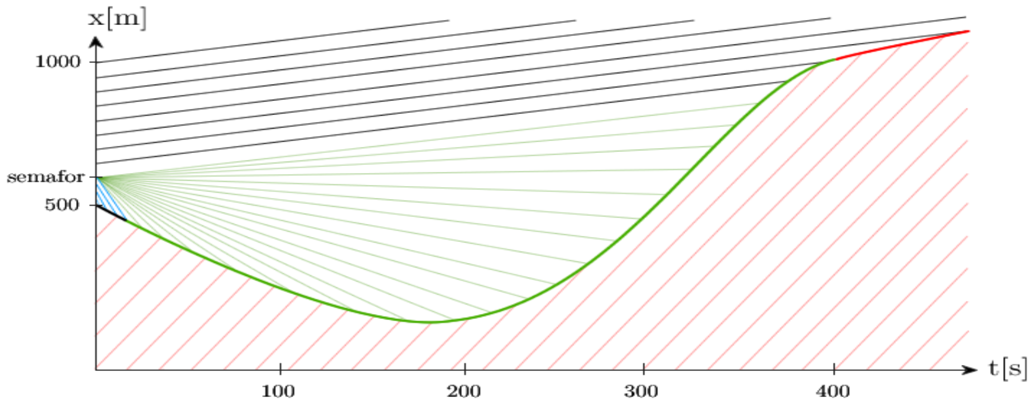


Fig. 12

The complete graph can determine the approximate time and place where there are changes in the flow of the traffic flow. At the point where the red characteristic intersects the green characteristic transferring the value of density approximately equal to 0.2 vehicle/m get in the sum, is the slope of transition line equals zero (this condition occurs in time approximately 180 s at the point approximately 160 m from the start). In the time about 360 s vehicles are already limited and can freely pass around a traffic light. The map of all characteristics given by the graph in fig. 12 was necessary to be downloaded in other width, because it would not be appropriately scaled in the text and it was necessary to draw the inverse characteristics to the coordinate system (t, x) (previous graphs were constructed into the coordinate system (x, t)) in order to get well arranged results. Therefore, slopes of characteristics quite precisely do not correspond to the reality, but the essence of the problem is captured.

Conclusion

The purpose of this paper was to suggest an expansion of preparation of learning materials designed for students of selected schools. In this work, we focused on solving easy traffic problem by mathematical modeling from a macroscopic perspective. Interpretation was presented in a way that should be easily grasped by students - university students who have chosen advanced mathematics as an elective, or high school students who are engaged in mathematics beyond compulsory curriculum

First, we derived a mathematical model in the form of partial differential equations. In order to apply the equation for an illustrative example of the practice it was be necessary to introduce and understand some important concepts such as the traffic density, the traffic jam, the continuous and reduced traffic or the flow function. Then we focused on solving this equation using the method of characteristics. The theoretical part is complemented by several practical examples describing situations encountered in everyday traffic.

Tento článok odporúčal na publikovanie vo vedeckom časopise Mladá veda:
doc. Vladimíra Petrášková, Ph.D.

References

1. DRÁBEK, Pavel, HOLUBOVÁ, Gabriela, 2007, Elements *of Partial Differential Equations*, Berlin : Walter de Gruyter. ISBN 978-3-11-019124-0.
2. HOKR, Milan, 2005. *Transportní procesy. Učební text,* Technická univerzita v Liberci.
3. JELÍNEK, Jiří, VYSOKÁ, J., 2013. *Modelování dynamiky dopravního proudu*, Silnice Železnice, Ostrava - Vítkovice: Konstrukce Media. ISSN 1801-822X.
4. JELÍNEK, Jiří, VYSOKÁ, Jana, 2013. *Porovnání modelů pro výpočet hustoty dopravního toku,* Silnice Železnice, Ostrava - Vítkovice: Konstrukce Media.110-112. ISSN 1801-822X.
5. JELÍNEK, Jiří, VYSOKÁ, Jana, 2014. *O podchodach v modelirovanii transportnoj dinamiki*, Vestnik Astrachanskogo gosudarstvennogo techničeskogo universiteta, Astrachan': Astrachanskij gosudarstvennyj techničeskij universitet. ISSN 2072-9502
6. KUTZLER, B., KOKOL-VOLJC, V., 2003. *Úvod do Derive 6*, OEG, Austria.
7. Le VEQUE R., 2002. *Finite Volume Methods for Hyperbolic Problems*, Cambridge University. ISBN 0521009243
8. Le VEQUE R., 2001. *Some Traffic Flow Models Illustrating Interesting Hyperbolic Behavior*.